

# NATURAL PARTIAL ORDER ON RINGS WITH INVOLUTION

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**ABSTRACT.** In this paper, we introduce a partial order on rings with involution, which is a generalization of the partial order on the set of projections in a Rickart  $*$ -ring. We prove that, a  $*$ -ring with the natural partial order form a sectionally semi-complemented poset. It is proved that every interval  $[0, x]$  forms an orthomodular lattice in case of abelian Rickart  $*$ -rings. The concepts of generalized comparability ( $GC$ ) and partial comparability ( $PC$ ) are extended to involve all the elements of a  $*$ -ring. Further, it is proved that these concepts are equivalent in finite abelian Rickart  $*$ -rings.

**Keywords :**  $*$ -ring, partial order, generalized comparability, partial comaparability.

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## 1. INTRODUCTION

An *involution*  $*$  on an associative ring  $R$  is a mapping such that  $(a+b)^* = a^* + b^*$ ,  $(ab)^* = b^*a^*$  and  $(a^*)^* = a$ , for all  $a, b \in R$ . A ring with involution  $*$  is called a  $*$ -ring. Clearly, identity mapping is an involution if and only if the ring is commutative. An element  $e$  of a  $*$ -ring  $R$  is a *projection* if  $e = e^2$  and  $e = e^*$ . For a nonempty subset  $B$  of  $R$ , we write  $r(B) = \{x \in R : bx = 0, \forall b \in B\}$ , and call a *right annihilator* of  $B$  in  $R$ . A *Rickart  $*$ -ring*  $R$  is a  $*$ -ring in which right annihilator of every element is generated, as a right ideal, by a projection in  $R$ . Every Rickart  $*$ -ring contains unity. For each element  $a$  in a Rickart  $*$ -ring, there is unique projection  $e$  such that  $ae = a$  and  $ax = 0$  if and only if  $ex = 0$ , called a *right projection* of  $a$  denoted by  $RP(a)$ . In fact,  $r(\{a\}) = (1 - RP(a))R$ . Similarly, the left annihilator  $l(\{a\})$  and the left projection  $LP(a)$  are defined for each element  $a$  in a Rickart  $*$ -ring  $R$ . The set of projections  $P(R)$  in a Rickart  $*$ -ring  $R$  forms a lattice, denoted by  $L(P(R))$ , under the partial order ' $e \leq f$  if and only if  $e = fe = ef$ '. In fact,  $e \vee f = f + RP(e(1 - f))$  and  $e \wedge f = e - LP(e(1 - f))$ . This lattice is extensively studied by I. Kaplanski [4], S. K. Berberian [1], S. Maeda in [5, 6] and others.

Drazin [2], was the first to introduce " $*$ -order" to involve all elements, where  $*$ -order is given by,  $a \leq_* b$  if and only if  $a^*a = a^*b$  and  $aa^* = ba^*$ , which is a partial order on a semigroup with proper involution (i.e.,  $aa^* = 0$  implies  $a = 0$ ). In particular, with the obvious choices for  $*$ -rings with proper involution, all commutative rings with no nonzero nilpotent elements, all Boolean rings, the ring  $B(H)$  of all bounded linear operators on any complex Hilbert space  $H$ , the Rickart  $*$ -ring. Janowitz [3] studies  $*$ -order on Rickart  $*$ -ring. Thakare and Nimbhorkar [9] used  $*$ -order on Rickart  $*$ -ring and generalized the comparability axioms to involve all elements of  $*$ -ring. Mitsch [8] defined a partial order on a semigroup. We modify that order to have partial order on a  $*$ -ring.

In this paper, we introduce a partial order on a  $*$ -ring which is a generalization of the partial order on the set of projections in a Rickart  $*$ -ring. For a  $*$ -ring  $R$ , it is proved that the poset  $(R, \leq)$  is an sectionally semi-complemented (SSC) poset. For an abelian Rickart  $*$ -ring, we prove that every interval  $[0, x]$  is an orthomodular poset, in fact, an orthomodular lattice. In the last section, comparability axioms are introduced to involve all elements of the  $*$ -ring.

## 2. NATURAL PARTIAL ORDER AND ITS PROPERTIES

We introduce an order on a  $*$ -ring with unity.

**Definition 2.1.** Let  $R$  be a  $*$ -ring with unity. Define a relation ' $\leq$ ' on  $R$  by ' $a \leq b$ ' if and only if  $a = xa = xb = ax^* = bx^*$ , for some  $x \in R$ .

**Proposition 2.2.** Let  $R$  be a  $*$ -ring with unity. Then the relation ' $\leq$ ' given in Definition 2.1 is a partial order on  $R$ .

*Proof.* Reflexive: for  $x = 1$ , we have  $a = xa = ax^*$ . Hence  $a \leq a, \forall a \in R$ .

Antisymmetric: Let  $a \leq b$  and  $b \leq a$ . Then there exist  $x, y \in R$  such that  $a = xa = xb = ax^* = bx^*$  and  $b = yb = ya = by^* = ay^*$ , hence  $b = ya = y(bx^*) = bx^* = a$ .

Transitive: Let  $a \leq b$  and  $b \leq c$ . Hence there exist  $x, y \in R$  such that  $a = xa = xb = ax^* = bx^*$  and  $b = yb = yc = by^* = cy^*$ . Then  $(xy)a = (xy)(bx^*) = x(yb)x^* = xbx^* = ax^* = a$ ,  $(xy)c = x(yc) = xb = a$ ,  $a(xy)^* = (xb)(y^*x^*) = x(by^*)x^* = xbx^* = ax^* = a$  and  $c(xy)^* = c(y^*x^*) = (cy^*)x^* = bx^* = a$ . Hence  $a \leq c$ .  $\square$

Henceforth  $R$  denotes a  $*$ -ring with unity and we say that  $a \leq b$  *through*  $x$  whenever  $a = xa = xb = ax^* = bx^*$ .

**Note 2.3.** If we restrict this partial order to the set of projections in a Rickart  $*$ -ring, then it coincides with the usual partial order for projections given in Berberian [1].

**Remark 2.4.** This order is different from  $*$ -order.

For, let  $a = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \in R = M_2(\mathbb{Z}_3)$  with transpose as an involution.

Then  $a^*a = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = a^*b$ ,  $aa^* = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = ba^*$ , hence  $a \leq_* b$ .

Next let  $x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$  be such that  $a = xa = xb = ax^* = bx^*$ . Then  $a = xa$  gives

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & 2(x_1 + x_2) \\ x_3 + x_4 & 2(x_3 + x_4) \end{bmatrix} \text{ and } a = ax^* \text{ gives}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 & x_3 + 2x_4 \\ x_1 + 2x_2 & x_3 + 2x_4 \end{bmatrix}.$$

On comparing, we get  $x_1 + x_2 = 1 = x_1 + 2x_2$ , which gives  $x_2 = 0, x_1 = 1$ . Similarly  $x_3 + x_4 = 1, x_3 + 2x_4 = 2$ , giving  $x_3 = 1, x_4 = 0$ , i.e.,  $x = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ . But  $xb \neq a$ .

Hence  $a \not\leq b$ . On the Other hand, if  $c = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ ,  $d = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$  and  $y = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ . Then  $c \leq d$  through  $y$ . While  $c^*c \neq c^*d$ , hence  $c \not\leq^* d$ . Thus these two partial orders (natural partial order and  $*$ -order) are distinct.

**Proposition 2.5.** *Let  $R$  be a commutative  $*$ -ring. Then  $a \leq b$  implies  $a \leq^* b$ .*

*Proof.* Let  $a \leq b$ . Then there exists  $x \in R$  such that  $a = xa = xb = ax^* = bx^*$ . This yields  $a^*b = (xa)^*b = a^*x^*b = a^*(bx^*) = a^*a$  and since  $R$  is commutative, we get  $aa^* = ba^*$ . Hence  $a \leq^* b$ .  $\square$

Note that the converse of the above statement is not true in general. Since  $*$ -order is not a partial order on  $\mathbb{Z}_{12}$  with identity mapping as an involution (as  $6^*6 = 6^*0 = 0^*0$ ).

In the next result, we provide properties of the natural partial order.

**Theorem 2.6.** *Let  $R$  be a  $*$ -ring with unity. Then the following statements hold.*

- (1)  $0$  is the least element of the poset  $R$ .
- (2)  $a \leq e, a \in R, e \in P(R)$  (set of projections in  $R$ ) implies  $a \in P(R)$ .
- (3)  $a \leq b$  if and only if  $a^* \leq b^*$ .
- (4) If  $a \leq b$ , then  $r(b) \subseteq r(a)$  and  $l(b) \subseteq l(a)$ .
- (5)  $a \leq b$ ,  $b$  regular (i.e.,  $bb'b = b$ , for some  $b' \in R$ ) implies  $a$  is regular.
- (6)  $a \leq b$  and  $a$  has right (resp. left) inverse imply  $a = b$ , i.e., every element having right (resp. left) inverse is maximal.
- (7) If  $a \leq b$ . Then  $ac \leq bc$  if and only if  $ca \leq cb, \forall c \in R$ .

*Proof.* (1) Obvious.

(2) Suppose  $a \leq e, e \in P(R)$ . Let  $a \leq e$  through  $x$ , for some  $x \in R$ , i.e.,  $a = xa = xe = ax^* = ex^*$ . This yields  $a^2 = xe.ex^* = xex^* = ax^* = a$ . Also,  $a^* = (xe)^* = ex^* = a$ , hence  $a \in P(R)$ .

(3) Let  $a \leq b$ . Then  $a = xa = xb = ax^* = bx^*$ , for some  $x \in R$ . Hence  $a^* = xa^* = a^*x^* = b^*x^* = xb^*$  which gives  $a^* \leq b^*$ . The Converse follows from the fact that  $(a^*)^* = a$ .

(4) Obvious.

(5) Suppose  $a \leq b$  and  $b$  is regular, i.e.,  $bb'b = b$ , for some  $b' \in R$ . Let  $a = xa = xb = ax^* = bx^*$ , for some  $x \in R$ . Then  $a = ax^* = xbx^* = xbb'bx^* = (xb)b'(bx^*) = ab'a$ . Hence  $a$  is regular.

(6) Let  $c \in R$  be such that  $ac = 1$  (resp.  $ca = 1$ ) and  $a \leq b$ . Let  $a = xa = xb = ax^* = bx^*$ , for some  $x \in R$ . Then  $a = xa$  (resp.  $a = ax^*$ ) gives  $ac = xac$  (resp.  $ca = cax^*$ ). Thus  $1 = x$  (resp.  $1 = x^*$ ). Hence  $a = b$ , i.e.,  $a$  is a maximal element.

(7) Suppose  $a \leq b$  and  $ac \leq bc, \forall c \in R$ . Since  $a \leq b$ , by (3) above, we have  $a^* \leq b^*$  giving  $a^*c^* \leq b^*c^*$ , which further yields  $(a^*c^*)^* \leq (b^*c^*)^*$ , i.e.,  $ca \leq cb$  and conversely.  $\square$

In a poset  $P$ , the *principal ideal* generated by  $a \in P$  is given by  $[a] = \{x \in P : x \leq a\}$ .

**Proposition 2.7.** *If  $a$  and  $b$  are central elements of a  $*$ -ring  $R$  which generate the same ideals of a ring  $R$ , then there is a order isomorphism between the set of elements  $\leq a$  and the set of elements  $\leq b$ .*

*Proof.* Let  $a$  and  $b$  are central elements with  $Ra = Rb$ . Then  $a = bs, b = at$ , for some  $s, t \in R$ . Denote  $(a] = \{x \in R : x \leq a\}$ . Define  $\phi : (a] \rightarrow (b]$  by  $\phi(x) = xt$ . We claim that  $xt \leq b, \forall x \in (a]$ . As  $x \leq a$ , we have  $x = x_1x = x_1a = ax_1^* = xx_1^*$ , for some  $x_1 \in R$ . Then  $x_1xt = xt, xtx_1^* = x_1atx_1^* = x_1bx_1^* = x_1x_1^*b = x_1x_1^*at = x_1ax_1^*t = x_1xt = xt, x_1b = x_1at = ax_1t = xt$  and  $bx_1^* = x_1^*b = x_1^*at = ax_1^*t = xt$ . Hence  $xt \leq b$ . Now, let  $x, y \in (a]$  be such that  $x = x_1x = x_1a = ax_1^* = xx_1^*$  and  $y = y_1y = y_1a = ay_1^* = yy_1^*$ , for some  $x_1, y_1 \in R$ . Then  $\phi(x) = \phi(y)$  if and only if  $xt = yt$  if and only if  $x_1at = y_1at$  if and only if  $x_1b = y_1b$  if and only if  $x_1a = y_1a$  if and only if  $x = y$ . Hence  $\phi$  is well defined and one-one. Let  $z \in (b]$ . Then as above  $zs \in (a]$  and  $z = z_1b = z_1z = bz_1^* = zz_1^*$ , for some  $z_1 \in R$ . Also  $\phi(zs) = zst = z_1bst = z_1at = z_1b = z$ , i.e.,  $\phi$  is a bijection.

Now, suppose that  $x, y \in (a]$  with  $x \leq y$ . Then  $x = x_1x = x_1a = ax_1^* = xx_1^*$ ,  $y = y_1y = y_1a = ay_1^* = yy_1^*$  and  $x = x_2x = x_2y = yx_2^* = xx_2^*$ , for some  $x_1, x_2, y_1 \in R$ . Next,  $(x_1x_2)xt = x_1xt = xt, (x_1x_2)yt = x_1xt = xt, xt(x_1x_2)^* = xtx_2^*x_1^* = x_1atx_2^*x_1^* = x_1bx_2^*x_1^* = x_1x_2^*x_1^*b = x_1x_2^*x_1^*at = ax_1x_2^*x_1^*t = xx_2^*x_1^*t = xx_1^*t = xt$  and  $yt(x_1x_2)^* = ytx_2^*x_1^* = y_1atx_2^*x_1^* = y_1bx_2^*x_1^* = y_1x_2^*x_1^*b = y_1x_2^*x_1^*at = y_1ax_2^*x_1^*t = yx_2^*x_1^*t = xx_1^*t = xt$ . Consequently  $\phi(x) \leq \phi(y)$ . Hence  $\phi$  is an order isomorphism. In fact,  $\psi : (b] \rightarrow (a]$  defined by  $\psi(y) = ys$ , works as an inverse of  $\phi$ .  $\square$

**Theorem 2.8** (Condition of Compatability). *If  $xa = ax^*, \forall a, x \in R$ , then the natural partial order is compatible with multiplication, i.e.,  $a \leq b$  implies  $ca \leq cb$ , for all  $c \in R$ .*

*Proof.* In view of Theorem 2.6 (7), it is enough to show that  $a \leq b$  implies  $ac \leq bc, \forall c \in R$ . Let  $a \leq b$ , then there exists  $x \in R$  such that  $a = xa = xb = ax^* = bx^*$ . Hence  $ac = xac = xbc = ax^*c = bx^*c$ , i.e.,  $ac = xac, acx^* = ca^*x^* = c(xa)^* = ca^* = ac$ . Also  $bcx^* = cb^*x^* = c(xb)^*ca^* = ac$ , hence  $ac \leq bc$ .  $\square$

**Definition 2.9.** Two elements  $a$  and  $b$  in a  $*$ -ring  $R$  are *orthogonal*, denoted by  $a \perp b$ , if there exists  $x \in R$  such that  $xa = a = ax^*$  and  $xb = 0 = bx^*$ .

The orthogonality relation in a  $*$ -ring has the following properties.

**Theorem 2.10.** *For elements  $a, b, c$  in a  $*$ -ring  $R$ , the following statements hold.*

- (1)  $a \perp a$  implies  $a = 0$ .
- (2)  $a \perp b$  if and only if  $b \perp a$  if and only if  $a \perp (-b)$ .
- (3)  $a \perp b, c \leq a$  imply  $c \perp b$ .
- (4)  $a \perp b$  if and only if  $a \leq a - b$ .
- (5)  $a \leq b$  implies  $b - a \leq b$  and  $b - a \perp a$ .
- (6) If  $a \perp b$ , then  $a \wedge b = 0$  and  $a + b$  is an upper bound of both  $a, b$ .
- (7)  $a \perp b, (a + b) \perp c$  imply  $a \perp (b + c)$ .

*Proof.* (1), (2) Obvious.

(3) Suppose that  $a \perp b$  and  $c \leq a$ . Let  $x, y \in R$  be such that  $a = xa = ax^*$ ,  $xb = 0 = bx^*$  and  $c = yc = cy^* = ya = ay^*$ . Then  $(yx)c = yxay^* = yay^* = cy^* = c$ . Similarly,  $c(yx)^* = c$ . On the other hand,  $(yx)b = 0$  and  $b(yx)^* = 0$ . Consequently,  $c \perp b$ .

(4) Suppose  $a$  and  $b$  are orthogonal. Let  $x \in R$  be such that  $a = xa = ax^*$  and  $xb = 0 = bx^*$ . Then  $a = x(a-b) = (a-b)x^* = xa = ax^*$ , hence  $a \leq a-b$ . Conversely, suppose that  $a \leq a-b$ . Let  $x \in R$  be such that  $a = x(a-b) = (a-b)x^* = xa = ax^*$ . Then  $a = x(a-b)$  and  $a = xa$  gives  $xb = 0$ . Similarly,  $bx^* = 0$ . Hence  $a \perp b$ .

(5) Let  $x \in R$  be such that  $a = xa = xb = ax^* = bx^*$ . Then  $(1-x)(b-a) = b-a-xb+xa = b-a-a+a = b-a$ ,  $(1-x)b = b-xb = b-a$ ,  $b(1-x)^* = b-bx^* = b-a$  and  $(b-a)(1-x)^* = b-a-bx^*-ax^* = b-a-a+a = b-a$ . Hence  $b-a \leq b$ . Also  $(1-x)(b-a) = b-a = (b-a)(1-x)^*$  and  $(1-x)a = 0 = a(1-x)^*$ . Hence  $b-a \perp a$ .

(6) Suppose  $a \perp b$  and  $x$  be such that  $xa = a = ax^*$ ,  $xb = 0 = bx^*$ . Let  $c \leq a$  and  $c \leq b$  i.e.  $c = x_1c = x_1a = cx_1^* = ax_1^*$  and  $c = x_2c = x_2b = cx_2^* = bx_2^*$ , for some  $x_1, x_2 \in R$ . Then  $x_1a = c = x_2b$  gives  $c = x_1a = x_2b = x_1ax^* = x_2bx^* = 0$ . Hence  $a \wedge b = 0$ . From (2) and (4), we have  $a \leq a+b$  and  $b = (a+b) - a \leq a+b$ .

(7) Suppose that  $a \perp b$ ,  $(a+b) \perp c$ . From (6), we have  $a \leq a+b$  and  $a+b \leq a+b+c$ . This gives  $a \leq a+b+c$ . Then from (5), we get  $b+c = (a+b+c) - a \leq a+b+c$  and  $(b+c) \perp a$ , as required.  $\square$

A poset  $P$  with 0 is called *sectionally semi-complemented* (in brief SSC) if, for  $a, b \in P$ ,  $a < b$ , there exists an element  $c \in P$  such that  $0 < c < b$  and  $\{a, c\}^l = \{0\}$ , where  $\{a, c\}^l = \{x \in P : x \leq a \text{ and } x \leq c\}$ . Thus from (5) and (6) of Theorem 2.10, we have the following result.

**Theorem 2.11.** *Let  $R$  be a  $*$ -ring. Then the poset  $(R, \leq)$  is an SSC poset.*

A ring is called an *abelian ring* if all of its idempotents are central.

**Proposition 2.12.** *In an abelian Rickart  $*$ -ring  $R$ , the following statements are equivalent.*

- i)  $a \leq b$ .
- ii) *There exists a projection  $e$  such that  $a = ae = be$ .*
- iii)  $ab = a^2 = ba$ .

*Proof.* i)  $\implies$  ii) Suppose  $a \leq b$ , then there exists  $x \in R$  such that  $a = xa = xb = ax^* = bx^*$ . Since  $a = xa$ , we have  $(1-x)a = 0$ . This gives  $a \in r\{1-x\} = eR$ , for some projection  $e \in R$ . Then  $ea = a = ae$  and  $(1-x)e = 0$ , i.e.,  $e = xe = ex^*$ . Also,  $a = xb$  implies  $ea = exb = xeb = eb$ . Thus  $a = ae = be$ .

ii)  $\implies$  iii) Obvious.

iii)  $\implies$  i) Let  $ab = a^2$ , i.e.,  $a(b-a) = 0$ . Then there exists a projection  $e$  such that  $a = ea = ea$  and  $e(b-a) = 0$ , i.e.,  $eb = ea = a$ , hence  $a \leq b$ .  $\square$

**Lemma 2.13.** *If  $R$  is an abelian Rickart  $*$ -ring, then  $a \perp b$  implies  $a \wedge b = 0$  and  $a \vee b = a + b$ .*

*Proof.* Suppose  $a \perp b$ . By Theorem 2.10 (6),  $a \wedge b = 0$  and  $a + b$  is an upper bound of  $a$  and  $b$ . Let  $a \leq c$  and  $b \leq c$ , then there exist projections  $e, f$  such that  $a = ea = ec$  and  $b = fb = fc$ . Since  $a \perp b$ , there exists  $x \in R$  such that  $xa = a = ax^*$  and  $xb = 0 = bx^*$ . Let  $y = ex + f(1-x)$ . Then  $y(a+b) = exa + exb + f(1-x)a + f(1-x)b = a+b$ ,  $(a+b)y^* = a+b$ ,  $yc = exc + f(1-x)c = a+b$  and  $cy^* = a+b$ , i.e.,  $a+b \leq c$ . Thus  $a \vee b = a+b$ .  $\square$

Before proceeding further, we need the definition of orthomodular poset. An *orthomodular poset* is a partially ordered set  $P$  with 0 and 1 equipped with a mapping  $x \rightarrow x^\perp$  (called the *orthocomplementation*) satisfying the conditions.

- i)  $a \leq b \Rightarrow b^\perp \leq a^\perp$ ,
- ii)  $(a^\perp)^\perp = a$  for all  $a \in P$ ,
- iii)  $a \vee a^\perp = 1$  and  $a \wedge a^\perp = 0$ , for all  $a \in P$ ,
- iv)  $a \leq b^\perp$  implies that  $a \vee b$  exists in  $P$ ,
- v)  $a \leq b \Rightarrow b = a \vee (a \vee b^\perp)^\perp$ .

The following result is essentially due to Marovt et al. [7, Theorem 1].

**Theorem 2.14.** *Let  $R$  be a Rickart  $*$ -ring. Then  $a \leqslant_* b$  if and only if there exist projections  $p$  and  $q$  such that  $a = pb = bq$ .*

Thus, from Proposition 2.12 and Theorem 2.14, the natural partial order and  $*$ -order are equivalent on abelian Rickart  $*$ -rings. This leads to the following two results which are also proved independently by Janowitz [3].

**Theorem 2.15.** *Let  $R$  is an abelian Rickart  $*$ -ring. Then every interval  $[0, x]$  is an orthomodular poset.*

We know that, if  $R$  is a Rickart  $*$ -ring, then the set of projection  $P(R)$  forms a lattice and the set  $\{e \in P(R) : e \leq x''\}$  is sub lattice of  $P(R)$ , where  $x'$  is a projection which generates the right annihilator of  $x$ .

**Theorem 2.16.** *In an abelian Rickart  $*$ -ring  $R$  every interval  $[0, x]$  is ortho-isomorphic to  $\{e \in P(R) : e \leq x''\}$ . Hence every interval  $[0, x]$  is an orthomodular lattice.*

### 3. COMPARABILITY AXIOMS

Two projections  $e$  and  $f$  are said to be *equivalent*, written  $e \sim f$ , if there is  $w \in eRf$  such that  $e = ww^*$  and  $f = w^*w$ , which is an equivalence relation on the set of projections in a Rickart  $*$ -ring. A projection  $e$  is said to be *dominated* by the projection  $f$ , denoted by  $e \lesssim f$ , if  $e \sim g \leq f$ , for some projection  $g$  in  $R$ . Two projections  $e$  and  $f$  are said to be *generalized comparable* if there exists a central projection  $h$  such that  $he \lesssim hf$  and  $(1-h)f \lesssim (1-h)e$ . A  $*$ -ring is said to satisfy the *generalized comparability (GC)* if any two projections are generalized comparable. Two projections  $e$  and  $f$  are said to be *partially comparable* if there

exist non zero projections  $e_0, f_0$  in  $R$  such that  $e_0 \leq e, f_0 \leq f$  and  $e_0 \sim f_0$ . If for any pair of projections in  $R$ ,  $eRf \neq 0$  implies  $e$  and  $f$  are partially comparable, then  $R$  is said to satisfy *partial comparability (PC)*. More about comparability axioms on the set of projections in a Rickart  $*$ -ring can be found in Berberian [1].

Drazin [2] extended the relation of equivalence of two projections to general elements of a  $*$ -ring as follows.

**Definition 3.1** ([2, Definition 2\*]). Let  $R$  be a  $*$ -ring with unity. We say that  $a \sim b$  if and only if there exists  $x \in aRb, y \in bRa$  such that  $aa^* = xx^*, bb^* = yy^*, a^*a = y^*y, b^*b = x^*x$ .

This relation is symmetric on a  $*$ -ring. Thakare and Nimbhorkar [9] extended the comparability axioms using the above relation and  $*$ -order to involve all elements of Rickart  $*$ -ring.

We provide a relation which is symmetric and transitive on general elements of  $*$ -ring as an extension of the relation of equivalence of two projections.

**Definition 3.2.** Let  $R$  be a  $*$ -ring with unity. We say that  $a \sim b$  if and only if there exists  $x, y \in R$  such that  $aa^* = xx^*, bb^* = yy^*, a^*a = y^*y, b^*b = x^*x$  with  $x = ax = xb$  and  $y = by = ya$ .

Now, we extend the concepts of dominance,  $GC, PC$  etc. from the set of projections in a Rickart  $*$ -ring to general elements in a  $*$ -ring.

**Definition 3.3.** (1) Let  $R$  be a  $*$ -ring with unity. We say that  $a$  is *dominated by*  $b$  if  $a \sim c \leq b$  for some  $c \in R$ . In notation  $a \lesssim b$ .  
 (2) A  $*$ -ring  $R$  is said to satisfy the *generalized comparability for elements (GC)* for elements, if for any  $a, b \in R$  there exists a central projection  $h$  such that  $ha \lesssim hb$  and  $(1-h)b \lesssim (1-h)a$ .  
 (3) Two elements  $a, b$  in a  $*$ -ring  $R$  are said to be *partially comparable* if there exists two non-zero elements  $c, d$  in  $R$  such that  $c \leq a, d \leq b$  with  $c \sim d$ . If for any  $a, b \in R$ ,  $aRb \neq 0$  implies  $a$  and  $b$  are partially comparable then we say that  $R$  has *partial comparability for elements (PC)*.

Clearly, if  $a \leq b$  or  $a \sim b$ , then  $a \lesssim b$ .

**Lemma 3.4.** If  $a \lesssim b$  and  $h$  is a central projection, then  $ha \lesssim hb$ .

**Definition 3.5.** Two elements  $a$  and  $b$  in a  $*$ -ring  $R$  are said to be *very orthogonal* if there exists a central projection  $h$  such that  $ha = a$  and  $hb = 0$ .

The relevance of very orthogonality to generalized comparability is as follows:

**Theorem 3.6.** If  $a$  and  $b$  are elements of a  $*$ -ring  $R$ . Then the following statements are equivalent.

- i)  $a$  and  $b$  are generalized comparable.
- ii) There exists orthogonal decompositions  $a = x + y, b = z + w$  with  $x \sim z, y$  and  $w$  are very orthogonal.

*Proof.* i)  $\Rightarrow$  ii) Suppose  $a$  and  $b$  are generalized comparable. Let  $h$  be a central projection such that  $ha \lesssim hb$  and  $(1-h)b \lesssim (1-h)a$ . Then  $ha \sim k_1 \leq hb$ ,  $(1-h)b \sim k_2 \leq (1-h)a$ , for some  $k_1, k_2 \in R$ . Hence  $k_1 = m_1 k_1 = m_1 hb = k_1 m_1^* = h b m_1^*$ , for some  $m_1 \in R$ . Then  $k_1 = m_1 hb$  gives  $k_1 h = m_1 h b h = m_1 h b = k_1$ . Similarly,  $k_2 = (1-h)k_2$ . Also  $h a k_2^* = h a (1-h)k_2^* = 0 = (ha)^* k_2$ ,  $(1-h)b k_1^* = [(1-h)b]^* k_1 = 0$ .

We claim that  $ha + k_2 \sim k_1 + (1-h)b$ . Since  $ha \sim k_1$ , there exist  $x_1, y_1 \in R$  such that  $(ha)(ha)^* = x_1 x_1^*$ ,  $k_1 k_1^* = y_1 y_1^*$ ,  $(ha)^*(ha) = y_1^* y_1$  and  $k_1^* k_1 = x_1^* x_1$  with  $x_1 = h a x_1 = x_1 k_1$  and  $y_1 = k_1 y_1 = y_1 h a$ . Clearly,  $x_1 = h x_1$  and  $y_1 = h y_1$ , since  $k_1 h = k_1$ . Similarly, Since  $(1-h)b \sim k_2$ , there exist  $x_2, y_2 \in R$  such that  $k_2 k_2^* = x_2 x_2^*$ ,  $[(1-h)b][(1-h)b]^* = y_2 y_2^*$ ,  $k_2^* k_2 = y_2^* y_2$  and  $[(1-h)b]^* [(1-h)b] = x_2^* x_2$  with  $x_2 = k_2 x_2 = x_2 (1-h)b$  and  $y_2 = (1-h)b y_2 = y_2 k_2$ . Clearly,  $x_2 = (1-h)x_2$  and  $y_2 = (1-h)y_2$ , since  $k_2(1-h) = k_2$ .

Let  $x = x_1 + x_2$  and  $y = y_1 + y_2$ . Since  $h k_2 = 0$  and  $(1-h)k_1 = 0$ , we have  $(ha + k_2)x = (ha + k_2)(x_1 + x_2) = h a x_1 + h a x_2 + k_2 x_1 + k_2 x_2 = x_1 + 0 + 0 + x_2 = x$ ,  $x[k_1 + (1-h)b] = (x_1 + x_2)[k_1 + (1-h)b] = x_1 k_1 + x_1(1-h)b + x_2 k_1 + x_2(1-h)b = x_1 + 0 + 0 + x_2 = x$ . Similarly, we have  $y = [k_1 + (1-h)b]y = y(ha + k_2)$ .

Also,  $xx^* = (x_1 + x_2)(x_1 + x_2)^* = x_1 x_1^* + x_1 x_2^* + x_2 x_1^* + x_2 x_2^* = x_1 x_1^* + 0 + 0 + x_2 x_2^* = (ha)(ha)^* + k_2 k_2^* = [ha + k_2][ha + k_2]^*$  and  $x^* x = (x_1 + x_2)^*(x_1 + x_2) = x_1^* x_1 + x_1^* x_2 + x_2^* x_1 + x_2^* x_2 = x_1^* x_1 + 0 + 0 + x_2^* x_2 = k_1^* k_1 + [(1-h)b]^* [(1-h)b] = [k_1 + (1-h)b]^* [k_1 + (1-h)b]$ . On the other hand,  $yy^* = (y_1 + y_2)(y_1 + y_2)^* = y_1 y_1^* + y_1 y_2^* + y_2 y_1^* + y_2 y_2^* = k_1 k_1^* + 0 + 0 + [(1-h)b][(1-h)b]^* = [k_1 + (1-h)b][k_1 + (1-h)b]^*$  and  $y^* y = (y_1 + y_2)^*(y_1 + y_2) = y_1^* y_1 + y_1^* y_2 + y_2^* y_1 + y_2^* y_2 = y_1^* y_1 + 0 + 0 + y_2^* y_2 = (ha)^*(ha) + k_2^* k_2 = [ha + k_2]^* [ha + k_2]$ . Therefore  $ha + k_2 \sim k_1 + (1-h)b$ .

Next, we claim that  $ha + k_2 \leq a$  and  $k_1 + (1-h)b \leq b$ . Since  $h$  is a central projection,  $k_2 \leq (1-h)a \leq a$  and  $ha \leq a$ , implies  $k_2 = x_1 k_2 = x_1 a = k_2 x_1^* = a x_1^*$  and  $ha = x_2 h a = x_2 a = h a x_2^* = a x_2^*$ , for some  $x_1, x_2 \in R$ . Let  $y_1 = x_1 + h x_2$ , then  $y_1(ha + k_2) = x_1 h a + x_1 k_2 + h x_2 h a + h x_2 k_2 = ha + k_2$ ,  $(ha + k_2)y_1^* = h a x_1^* + k_2 x_1^* + h a x_2^* h + k_2 x_2^* h = ha + k_2$  and  $y_1 a = a(x_1 + h x_2)^* = ha + k_2 = a y_1^*$ , therefore  $ha + k_2 \leq a$ . Similarly,  $(1-h)b + k_1 \leq b$ . Now put  $ha + k_2 = x$ ,  $(1-h)b + k_1 = z$ ,  $y = a - x$  and  $w = b - z$  implies  $h b - k_1 = b - z = w$ . Then  $h w = h(h b - k_1) = h b - k_1 = w$  and  $h y = h(a - x) = ha - h x = ha - ha - h k_2 = 0$  (since  $h k_2 = 0$ ), i.e.,  $y$  and  $w$  are very orthogonal. Thus  $a = x + y$ ,  $b = z + w$  where  $x \perp y$ ,  $z \perp w$  such that we get  $x \sim z$  with  $y$  and  $w$  are very orthogonal.

ii)  $\Rightarrow$  i) Let  $h$  be a central projection such that  $h w = w$  and  $h y = 0$ . Then  $ha = h x + h y = h x$  and  $(1-h)b = (1-h)z + (1-h)w = (1-h)z$ , where  $ha = h x \sim h z \leq hb$  and  $(1-h)b = (1-h)z \sim (1-h)x \leq (1-h)a$ . Thus  $ha \lesssim hb$  and  $(1-h)b \lesssim (1-h)a$ . Hence  $a, b$  are generalized comparable.  $\square$

Next result implies that  $GC$  for elements is stronger than  $PC$  for elements.

**Theorem 3.7.** *If  $R$  is a  $*$ -ring with  $GC$  for elements then it has  $PC$  for elements.*

*Proof.* Let  $a, b$  are elements of  $R$  which are not partially comparable. We will show that  $a R b = 0$ . Applying  $GC$  to the pair  $a, b$  we get orthogonal decompositions  $a = x + y$  and  $b = z + w$ , where  $x \sim z$  and  $y, w$  are very orthogonal. If  $x \neq 0$



and  $w \neq 0$  then  $a$  and  $b$  are partially comparable, which is a contradiction to the assumption. Hence  $x = 0 = w$ , i.e.,  $a, b$  are very orthogonal. Let  $h$  be a central projection such that  $ha = a$  and  $hb = 0$ . Then  $aRb = haRb = aRh b = 0$ . Thus  $R$  has  $PC$  for elements.  $\square$

**Lemma 3.8.** *In an abelian Rickart  $*$ -ring  $a \perp b$  if and only if  $RP(a)RP(b) = 0$ .*

*Proof.* First we show that  $ab = 0$  if and only if  $RP(a)RP(b) = 0$ . Suppose that  $ab = 0$  which gives  $b \in r(\{a\}) = (1 - RP(a))R$ . Hence  $(1 - RP(a))b = b$  giving  $RP(a)b = 0$ . Since all projections in  $R$  are central, we get  $RP(a) \in r(\{b\}) = (1 - RP(b))R$ . Which yields  $RP(b)RP(a) = 0$ . Conversely, if  $RP(a)RP(b) = 0$ , then  $ab = (aRP(a))(bRP(b)) = aRP(a)RP(b)b = 0$ .

Next, Suppose that  $a \perp b$ . Then there exists  $x \in R$  such that  $xa = a = ax^*$  and  $xb = 0 = bx^*$ , i.e.,  $a(1 - x^*) = 0$ . Hence  $RP(a)RP(1 - x^*) = 0$ . Since  $R$  is abelian, we have  $RP(1 - x^*) = 1 - RP(x^*) = 1 - RP(x)$ . Consequently,  $RP(a)RP(x) = RP(a)$ . On the other hand,  $xb = 0$  implies  $RP(x)RP(b) = 0$ . Then  $RP(a)RP(b) = RP(a)RP(x)RP(b) = 0$ , hence  $ab = 0$ . Conversely, if  $ab = 0$ , then  $RP(a)RP(b) = 0$ . Thus  $RP(a)a = a = aRP(a)$  and  $RP(a)b = 0 = bRP(a)$ . Hence  $a \perp b$ .  $\square$

The next result shows that the relation  $\sim$  is finitely additive.

**Theorem 3.9.** *Let  $R$  be an abelian Rickart  $*$ -ring. If  $a_1 \perp a_2$ ,  $b_1 \perp b_2$  with  $a_1 \sim b_1$  and  $a_2 \sim b_2$ , then  $a_1 + a_2 \sim b_1 + b_2$ , i.e., the relation  $\sim$  is finitely additive.*

*Proof.* Since  $a_1 \perp a_2$ ,  $b_1 \perp b_2$ , we have  $RP(a_1)RP(a_2) = 0 = RP(b_1)RP(b_2)$ . Also,  $a_1 \sim b_1$  and  $a_2 \sim b_2$  there exists  $x_i, y_i \in R$  such that  $a_i a_i^* = x_i x_i^*$ ,  $a_i^* a_i = y_i^* y_i$ ,  $b_i b_i^* = y_i y_i^*$ ,  $b_i^* b_i = x_i x_i$  with  $x_i = a_i x_i = x_i b_i$  and  $y_i = b_i y_i = y_i a_i$  for  $i = 1, 2$ . This gives  $x_i(1 - a_i) = 0$  (since in an abelian Rickart  $*$ -ring  $RP(x) = LP(x)$ ), hence  $RP(x_i) = RP(x_i)RP(a_i)$ , for  $i = 1, 2$ . Then for  $i \neq j$ , we have  $x_i a_j = x_i RP(x_i) a_j RP(a_j) = x_i RP(x_i) RP(a_i) a_j RP(a_j) = x_i RP(x_i) RP(a_1) RP(a_j) a_j = 0$ . Moreover  $x_i x_j^* = 0 = x_i^* x_j$  for  $i \neq j$ . Similarly, we have  $b_j x_i = 0$  for  $i \neq j$ .

Let  $x = x_1 + x_2$  and  $y = y_1 + y_2$ . Then  $(a_1 + a_2)x = a_1 x_1 + a_2 x_1 + a_1 x_2 + a_2 x_2 = x_1 + 0 + 0 + x_2$  and  $(b_1 + b_2)x = b_1 x_1 + b_1 x_2 + b_2 x_1 + b_2 x_2 = x_1 + 0 + 0 + x_2 = x$ . Consider  $xx^* = x_1 x_1^* + x_2 x_1^* + x_1 x_2^* + x_2 x_2^* = a_1 a_1^* + 0 + 0 + a_2 a_2^* = (a_1 + a_2)(a_1 + a_2)^*$  and  $x^* x = x_1^* x_1 + x_2^* x_1 + x_1^* x_2 + x_2^* x_2 = b_1^* b_1 + b_1^* b_2 = (b_1 + b_2)^*(b_1 + b_2)$ . Similarly,  $y = (b_1 + b_2)y = y(a_1 + a_2)$ ,  $yy^* = (b_1 + b_2)(b_1 + b_2)^*$  and  $y^* y = (a_1 + a_2)^*(a_1 + a_2)$ . Therefore  $a_1 + a_2 \sim b_1 + b_2$ .  $\square$

Above result ensures that the converse of Theorem 3.7 is true for finite abelian Rickart  $*$ -rings.

**Theorem 3.10.** *Let  $R$  be a finite abelian Rickart  $*$ -ring. Then  $GC$  for elements and  $PC$  for elements are equivalent.*

*Proof.* Suppose that  $R$  has  $PC$  for elements. It is enough to show that,  $PC$  for elements implies  $GC$  for elements. Let  $a, b \in R$ . If  $aRb = 0$ , then  $ab = 0$ . This

gives  $RP(a)b = 0$ . Since  $R$  is an abelian ring, we get  $a$  and  $b$  are very orthogonal. Hence we are done. Suppose  $aRb \neq 0$ . Hence there exist  $a_0 \leq a$  and  $b_0 \leq b$  such that  $a_0 \sim b_0$ . Let  $a_1, b_1$  be the largest elements such that  $a_1 \leq a$ ,  $b_1 \leq b$  and  $a_1 \sim b_1$ . Then  $a_2 = a - a_1$  and  $b_2 = b - b_1$  are such that  $a_2 \leq a$ ,  $b_2 \leq b$ ,  $a_1 \perp a_2$  and  $b_1 \perp b_2$ . By the maximality of  $a_1$  and  $b_1$ , we get  $a_2Rb_2 = 0$ , which gives  $a_2$  and  $b_2$  very orthogonal. Thus we get an orthogonal decompositions  $a = a_1 + a_2$ ,  $b = b_1 + b_2$  such that  $a_1 \sim b_1$ ,  $a_2$  and  $b_2$  very orthogonal. By Theorem 3.6 we have  $a$  and  $b$  are generalized comparable.  $\square$

**Proposition 3.11.** *Let  $R$  be a  $*$ -ring with  $GC$  for elements and  $e$  is any projection in  $R$ . Then  $eRe$  also has  $GC$  for elements.*

*Proof.* Let  $a, b \in eRe \subseteq R$ . Then there exists a central projection  $h$  in  $R$  such that  $ha \lesssim hb$ ,  $(1 - h)b \lesssim (1 - h)a$ . Let  $g = ehe = he \in eRe$  and  $x$  be any element in  $eRe$ . Then  $gx = hex = hx = xh = xeh = xhe = xg$ . Hence  $g$  is a central projection in  $eRe$  with  $ga = hea = ha$ ,  $gb = heb = hb$ , i.e.,  $ga \lesssim gb$  and  $(e - g)b = ab = hab = b - hb = (1 - h)b$ ,  $(e - g)a = ea - hea = a - ha = (1 - h)a$ , i.e.,  $(e - g)b \lesssim (e - g)a$ . Thus  $a$  and  $b$  are generalized comparable in  $eRe$ .  $\square$

**Corollary 3.12.** *If the matrix ring  $M_n(R)$  has  $GC$  for elements, then  $R$  has  $GC$  for elements.*

An ideal  $I$  of a  $*$ -ring  $R$  is a  $*$ -ideal if  $a^* \in I$  whenever  $a \in I$ .

**Proposition 3.13.** *Let  $I$  be a  $*$ -ideal of  $R$ . If  $R$  has  $GC$  for elements, then  $R/I$  has  $GC$  for elements.*

*Proof.* Let  $a + I, b + I \in R/I$ . Applying  $GC$  to  $a, b \in R$ , there exists a central projection  $h \in R$  such that  $ha \lesssim hb$  and  $(1 - h)b \lesssim (1 - h)a$ . Then passing to cosets,  $h + I$  is central projection in  $R/I$  such that  $(h + I)(a + I) \lesssim (h + I)(b + I)$  and  $[(1 + I) - (h + I)](b + I) \lesssim [(1 + I) - (h + I)](a + I)$ . Hence  $R/I$  has  $GC$  for elements.  $\square$

**Remark 3.14.** The converse of above statement is not true. For, let  $R = \mathbb{Z}_{10}$  with identity map as an involution and  $I = \{0, 2, 4, 6, 8\}$ . Then  $R/I = \{0 + I, 1 + I\}$  which has  $GC$  for elements trivially. The poset  $R$  with natural partial order is depicted in Figure 1.

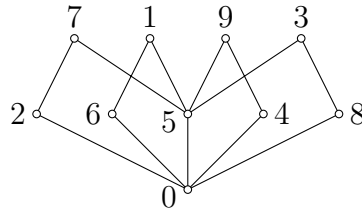


Figure 1

Here  $R$  does not have  $GC$  for elements. On the contrary, if  $R$  has  $GC$  for elements, then by Theorem 3.7,  $R$  has  $PC$  for elements. Let  $a = 2$  and  $b = 4$ . Then  $aRb \neq 0$

and  $2 \approx 4$ , Since  $22^* = 4$  and  $4^*4 = 6$  and  $R$  being commutative there is no  $x \in R$  such that  $xx^* = 4$  and  $x^*x = 6$ . Hence 2 and 4 are not partially comparable in  $R$ , a contradiction.

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